

TSIRELSON-LIKE SPACES AND COMPLEXITY OF CLASSES OF BANACH SPACES

ONDŘEJ KURKA

ABSTRACT. Employing a construction of Tsirelson-like spaces due to Argyros and Deliyanni, we show that the class of all Banach spaces which are isomorphic to a subspace of c_0 is a complete analytic set with respect to the Effros Borel structure of separable Banach spaces. Moreover, the classes of all separable spaces with the Schur property and of all separable spaces with the Dunford-Pettis property are Π_2^1 -complete.

1. INTRODUCTION AND MAIN RESULTS

During the last two decades, it turned out that descriptive set theory provides a fruitful approach to several questions in separable Banach space theory. A particular and generally still not well understood question is the question of the descriptive complexity of a given class of separable Banach spaces. In the present work, we introduce a new approach to complexity problems in Banach space theory which is based on a fundamental example of Tsirelson.

The connections between descriptive set theory and Banach space theory were discovered by J. Bourgain [4, 5]. Later, B. Bossard [3] investigated codings of separable Banach spaces up to isomorphism by standard Borel spaces and used the Effros Borel structure for studying complexity questions in Banach space theory (see Section 2 for the definitions of the Effros Borel structure and of the related notions used below, let us note here that by an isomorphism we mean a linear isomorphism throughout this paper).

It can be shown quite easily that the isomorphism class of any separable Banach space is analytic. B. Bossard asked in [3] whether ℓ_2 is

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(up to isomorphism) the only infinite-dimensional separable Banach space whose isomorphism class is Borel. There are several examples for which the isomorphism class is shown to be non-Borel, for instance Pełczyński's universal space [3], $C(2^{\mathbb{N}})$ (see e.g. [18, (33.26)]) or $L_p([0, 1])$ for $1 < p < \infty, p \neq 2$, (see e.g. [9]). A by-product of the present work are two new examples $(\bigoplus G_n)_{c_0}$ and $(\bigoplus G_n)_{\ell_1}$ (see Remarks 3.9(ii) and 3.10(vii)).

Bossard's question has been recently answered by G. Godefroy [11] who has proven the existence of a space which is not isomorphic to ℓ_2 but the isomorphism class of which is Borel. The following question posed in [10], however, remains open.

Question 1.1 (Godefroy). *Is the class of all Banach spaces isomorphic to c_0 Borel?*

In Section 7, we present some remarks concerning this interesting problem. Although we have not found its solution, we have obtained the following related result.

Theorem 1.2. *The class of all Banach spaces which can be embedded isomorphically into c_0 is complete analytic. In particular, it is not Borel.*

This result answers [10, Problem 4] and provides most likely the first example of a space X for which the class of spaces embeddable into X is shown not to be Borel. In other words, we have proven that the embeddability relation $Y \hookrightarrow X$ has a non-Borel horizontal section $\cdot \hookrightarrow X$. This discovery is not surprising, as the vertical section $Y \hookrightarrow \cdot$ is known to be non-Borel for every infinite-dimensional Y (see [3, Corollary 3.3(vi)]).

Our second main result is based on a combination of methods used for proving Theorem 1.2 with a tree space method used in [20].

Theorem 1.3. *The classes of all separable Banach spaces with the Schur property and of all separable Banach spaces with the Dunford-Pettis property are Π_2^1 -complete. In particular, these classes are not Σ_2^1 .*

This result answers two questions posed by B. M. Braga in [6]. We recall that a Banach space X is said to have the *Schur property* if every weakly convergent sequence in X is norm convergent. The *Dunford-Pettis property* is defined in Section 4. We note here just that a remarkable characterization states that X has the Dunford-Pettis property if and only if $x_n^*(x_n) \rightarrow x^*(x)$ whenever $x_n^* \rightarrow x^*$ weakly in X^* and $x_n \rightarrow x$ weakly in X .

Both results above are significantly based on a construction due to S. A. Argyros and I. Deliyanni [1] who generalized the well-known

example of B. S. Tsirelson [22]. Let us recall the definition of this important example.

For $E \subset \mathbb{N}$ and $x \in c_{00}(\mathbb{N})$, we denote by Ex the restriction of x on E , i.e., the element of $c_{00}(\mathbb{N})$ given by $Ex(i) = x(i)$ for $i \in E$ and $Ex(i) = 0$ for $i \notin E$. A family $\{E_1, \dots, E_n\}$ of successive finite subsets of \mathbb{N} is said to be *admissible* if

$$n < E_1 < E_2 < \dots < E_n.$$

The system of all admissible families is denoted by adm .

Definition 1.4 (Tsirelson). Let Θ be the smallest absolutely convex subset of $c_{00}(\mathbb{N})$ containing every basic vector $e_i = \mathbf{1}_{\{i\}}$, $i \in \mathbb{N}$, and satisfying

$$\{E_1, \dots, E_n\} \in \text{adm} \ \& \ x_1, \dots, x_n \in \Theta \quad \Rightarrow \quad \frac{1}{2} \sum_{k=1}^n E_k x_k \in \Theta.$$

Let $\|\cdot\|_{\mathcal{T}^*}$ be the Minkowski gauge of Θ and let \mathcal{T}^* be a completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{\mathcal{T}^*})$.

The space \mathcal{T}^* is the first example of an infinite-dimensional Banach space not containing an isomorphic copy of c_0 or any ℓ_p . It is well-known that \mathcal{T}^* is reflexive and dual to the space \mathcal{T} defined as the Banach space of sequences $x = \{x(i)\}_{i=1}^\infty$ with the basis $e_i = \mathbf{1}_{\{i\}}$ and with the implicitly defined norm

$$\|x\|_{\mathcal{T}} = \max \left\{ \|x\|_\infty, \frac{1}{2} \sup \left\{ \sum_{k=1}^n \|E_k x\|_{\mathcal{T}} : \{E_1, \dots, E_n\} \in \text{adm} \right\} \right\}.$$

2. PRELIMINARIES I

Our terminology concerning Banach space theory and descriptive set theory follows [8] and [18].

A *Polish space (topology)* means a separable completely metrizable space (topology). A set X equipped with a σ -algebra is called a *standard Borel space* if the σ -algebra is generated by a Polish topology on X .

A subset A of a standard Borel space X is called an *analytic set* (or a Σ_1^1 set) if there exist a standard Borel space Y and a Borel subset B of $X \times Y$ such that A is the projection of B on the first coordinate. The complement of an analytic set is called a *coanalytic set* (or a Π_1^1 set).

A subset A of a standard Borel space X is called a Σ_2^1 set if there exist a standard Borel space Y and a coanalytic subset B of $X \times Y$ such

that A is the projection of B on the first coordinate. The complement of a Σ_2^1 set is called a Π_2^1 set.

Let Γ be a class of sets in standard Borel spaces (for example Σ_1^1 or Π_2^1). A subset A of a standard Borel space X is called a Γ -hard set if every Γ subset B of a standard Borel space Y admits a Borel mapping $f : Y \rightarrow X$ such that $f^{-1}(A) = B$. A subset A of a standard Borel space X is called a Γ -complete set if it is Γ and Γ -hard at the same time.

A Σ_1^1 -hard (Σ_1^1 -complete, Π_1^1 -hard, Π_1^1 -complete) set may be called also *hard analytic* (*complete analytic*, *hard coanalytic*, *complete coanalytic*).

We note that the introduced notion of a hard (complete) set is suitable for classes like Σ_1^1 or Π_2^1 but not for Borel classes in Polish spaces. In that case, only a zero-dimensional Y and a continuous f are considered.

Let us recall a standard simple argument for Γ -hardness of a set.

Lemma 2.1. *Let $A \subset X$ and $C \subset Z$ be subsets of standard Borel spaces X and Z . Assume that C is Γ -hard. If there is a Borel mapping $g : Z \rightarrow X$ such that*

$$g(z) \in A \iff z \in C,$$

then A is Γ -hard as well.

For a topological space X , we denote by $\mathcal{F}(X)$ the family of all closed subsets of X and by $\mathcal{K}(X)$ the family of all compact subsets of X .

The *hyperspace of compact subsets of X* is defined as $\mathcal{K}(X)$ equipped with the *Vietoris topology*, i.e., the topology generated by the sets of the form

$$\begin{aligned} \{K \in \mathcal{K}(X) : K \subset U\}, \\ \{K \in \mathcal{K}(X) : K \cap U \neq \emptyset\}, \end{aligned}$$

where U varies over open subsets of X . If X is Polish, then so is $\mathcal{K}(X)$.

We will need the following classical result (see e.g. [18, (27.4)]).

Theorem 2.2 (Hurewicz). *If X is Polish and $D \subset X$ is G_δ but not F_σ , then $\{K \in \mathcal{K}(X) : K \cap D \neq \emptyset\}$ is complete analytic.*

The set $\mathcal{F}(X)$ of all closed subsets of X can be equipped with the *Effros Borel structure*, defined as the σ -algebra generated by the sets

$$\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\},$$

where U varies over open subsets of X . If X is Polish, then, equipped with this σ -algebra, $\mathcal{F}(X)$ forms a standard Borel space.

It is well-known that the space $C([0, 1])$ contains an isometric copy of every separable Banach space. By the *standard Borel space of separable Banach spaces* we mean

$$\mathcal{SE}(C([0, 1])) = \{F \in \mathcal{F}(C([0, 1])) : F \text{ is linear}\},$$

considered as a subspace of $\mathcal{F}(C([0, 1]))$.

Whenever we say that a class of separable Banach spaces has a property like being analytic, complete analytic, Π_2^1 -complete etc., we consider the class as a subset of $\mathcal{SE}(C([0, 1]))$.

By $c_{00}(\Lambda)$ we denote the vector space of all systems $x = \{x(\lambda)\}_{\lambda \in \Lambda}$ of scalars such that $x(\lambda) = 0$ for all but finitely many λ 's. By the canonical basis of $c_{00}(\Lambda)$ we mean the algebraic basis consisting of vectors $\mathbf{1}_{\{\lambda\}}$, $\lambda \in \Lambda$. Instead of $c_{00}(\mathbb{N})$, we write simply c_{00} .

In the context of Banach spaces, by a basis we mean a Schauder basis. A basis $\{x_i\}_{i=1}^\infty$ of a Banach space X is said to be *1-unconditional* if $\|\sum_{i \in A} a_i x_i\| \leq \|\sum_{i \in B} a_i x_i\|$ whenever $A \subset B$ are finite sets of natural numbers and $a_i \in \mathbb{R}$ for $i \in B$.

A basis $\{x_i\}_{i=1}^\infty$ of a Banach space X is said to be *shrinking* if

$$X^* = \overline{\text{span}}\{x_1^*, x_2^*, \dots\}$$

where x_1^*, x_2^*, \dots is the dual basic sequence $x_n^* : \sum_{i=1}^\infty a_i x_i \mapsto a_n$. The basis $\{x_i\}_{i=1}^\infty$ is called *boundedly complete* if $\sum_{i=1}^\infty a_i x_i$ is convergent whenever the sequence of its partial sums is bounded.

Let us recall a classical criterion of reflexivity (see e.g. [8, Theorem 6.11]).

Theorem 2.3 (James). *Let X be a Banach space with a basis $\{x_i\}_{i=1}^\infty$. Then X is reflexive if and only if $\{x_i\}_{i=1}^\infty$ is shrinking and boundedly complete.*

The remainder of this section is devoted to the proof of the following preliminary result.

Lemma 2.4. *Let Ξ be a standard Borel space and let $\{(X_\xi, \|\cdot\|_\xi)\}_{\xi \in \Xi}$ be a system of Banach spaces each member of which contains a sequence x_1^ξ, x_2^ξ, \dots whose linear span is dense in X_ξ . Assume that the function*

$$\xi \mapsto \left\| \sum_{k=1}^n \lambda_k x_k^\xi \right\|_\xi$$

is Borel whenever $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Then there exists a Borel mapping $\mathfrak{S} : \Xi \rightarrow \mathcal{SE}(C([0, 1]))$ such that $\mathfrak{S}(\xi)$ is isometric to X_ξ for every $\xi \in \Xi$.

We need to recall some definitions first. Let $\varepsilon > 0$ and let X, Y be Banach spaces. A linear operator $f : X \rightarrow Y$ is called an ε -isometry if

$$(1 + \varepsilon)^{-1} \|x\| < \|f(x)\| < (1 + \varepsilon) \|x\|, \quad x \in X \setminus \{0\}.$$

A separable Banach space \mathbb{G} is called *Gurariy* if, for every $\varepsilon > 0$, every finite-dimensional Banach spaces X and Y with $X \subset Y$ and every isometry $f : X \rightarrow \mathbb{G}$, there exists some ε -isometry $g : Y \rightarrow \mathbb{G}$ which extends f . It is known that there exists only one Gurariy space up to isometry ([21], see also [19]).

Lemma 2.5 (Kubiś, Solecki). *Let X_0 and X_1 be finite-dimensional Banach spaces with $X_0 \subset X_1$ and let $f : X_0 \rightarrow \mathbb{G}$ be a 2^{-n} -isometry. Then there is a $2^{-(n+1)}$ -isometry $g : X_1 \rightarrow \mathbb{G}$ such that $\|g|_{X_0} - f\| < 2 \cdot 2^{-n}$.*

This lemma is proven in [19] and its purpose is to show that \mathbb{G} contains an isometric copy of every separable Banach space X . Actually, once the lemma is proven, an isometry $f : X \rightarrow \mathbb{G}$ can be found easily. Let x_1, x_2, \dots be a dense sequence in X and let $X_n = \text{span}\{x_1, \dots, x_n\}$. Then Lemma 2.5 allows us to construct a sequence of linear operators $f_n : X_n \rightarrow \mathbb{G}$ such that f_n is a 2^{-n} -isometry and $\|f_{n+1}|_{X_n} - f_n\| < 2 \cdot 2^{-n}$. For $x \in X_m$, the sequence $\{f_n(x)\}_{n \geq m}$ is Cauchy. The isometry $f_\infty(x) = \lim_{n \rightarrow \infty} f_n(x)$ can be extended from $\bigcup_{m=1}^\infty X_m$ to an isometry $f : X \rightarrow \mathbb{G}$.

To prove Lemma 2.4, we use a leftmost branch argument to show that this construction can be accomplished in a Borel measurable way.

Proof of Lemma 2.4. We establish two additional assumptions which make the situation a bit simpler.

(1) We assume that all spaces X_ξ are infinite-dimensional. This is possible, because Ξ can be decomposed into Borel sets $\Xi_d = \{\xi \in \Xi : \dim X_\xi = d\}$ where $0 \leq d \leq \infty$. As these sets are Borel, we can deal with every Ξ_d separately. We consider only Ξ_∞ since other Ξ_d 's can be handled in a similar way.

(2) We assume moreover that x_n^ξ does not belong to the linear span of $x_1^\xi, \dots, x_{n-1}^\xi$. This is possible, because the sets

$$\{\xi \in \Xi : x_n^\xi \in \text{span}\{x_1^\xi, \dots, x_{n-1}^\xi\}\}, \quad n \in \mathbb{N},$$

are Borel, and so omitting the members which are a linear combination of its predecessors does not disrupt the assumption of the lemma (the resulting sequence will be infinite due to the first additional assumption).

Now, for every $n \in \mathbb{N}$, let $u_{n,1}, u_{n,2}, \dots$ be a sequence which is dense in the space $\mathcal{L}(\mathbb{R}^n, \mathbb{G})$ of linear operators from \mathbb{R}^n into \mathbb{G} . Let us define

$$X_n^\xi = \text{span}\{x_1^\xi, \dots, x_n^\xi\}$$

and consider operators

$$u_{n,i}^\xi : X_n^\xi \rightarrow \mathbb{G}, \quad u_{n,i}^\xi \left(\sum_{k=1}^n \lambda_k x_k^\xi \right) = u_{n,i}(\lambda_1, \dots, \lambda_n).$$

The operators are well defined since we assume that x_1^ξ, \dots, x_n^ξ are linearly independent. Notice that every $u \in \mathcal{L}(X_n^\xi, \mathbb{G})$ can be approximated by some $u_{n,i}^\xi$ with an arbitrarily small error. Therefore, we obtain for every $\xi \in \Xi$ that

- there exists $j \in \mathbb{N}$ such that $u_{1,j}^\xi : X_1^\xi \rightarrow \mathbb{G}$ is a 2^{-1} -isometry,
- if $u_{n,i}^\xi : X_n^\xi \rightarrow \mathbb{G}$ is a 2^{-n} -isometry, then Lemma 2.5 provides $j \in \mathbb{N}$ such that $u_{n+1,j}^\xi : X_{n+1}^\xi \rightarrow \mathbb{G}$ is a $2^{-(n+1)}$ -isometry and $\|u_{n+1,j}^\xi|_{X_n^\xi} - u_{n,i}^\xi\| < 2 \cdot 2^{-n}$.

Using the assumption of the lemma, it is straightforward to show that the sets

$$\begin{aligned} A_j &= \{\xi \in \Xi : u_{1,j}^\xi \text{ is a } 2^{-1}\text{-isometry}\}, \\ B_{n,i,j} &= \{\xi \in \Xi : u_{n+1,j}^\xi \text{ is a } 2^{-(n+1)}\text{-isometry} \\ &\quad \text{and } \|u_{n+1,j}^\xi|_{X_n^\xi} - u_{n,i}^\xi\| < 2 \cdot 2^{-n}\}, \end{aligned}$$

are Borel for every $n, i, j \in \mathbb{N}$.

We define recursively Borel functions $j_n : \Xi \rightarrow \mathbb{N}, n = 1, 2, \dots$, as follows. These functions are required to satisfy

- (i) $u_{n,j_n(\xi)}^\xi : X_n^\xi \rightarrow \mathbb{G}$ is a 2^{-n} -isometry,
- (ii) $\|u_{n+1,j_{n+1}(\xi)}^\xi|_{X_n^\xi} - u_{n,j_n(\xi)}^\xi\| < 2 \cdot 2^{-n}$.

Let $j_1(\xi)$ be the least natural number j such that $u_{1,j}^\xi : X_1^\xi \rightarrow \mathbb{G}$ is a 2^{-1} -isometry. We already know that such a number exists. The function j_1 is Borel, as

$$(j_1)^{-1}(\{j\}) = A_j \setminus \bigcup_{l=1}^{j-1} A_l.$$

Assuming that $j_n(\xi)$ is defined, let $j_{n+1}(\xi)$ be the least natural number j such that $u_{n+1,j}^\xi : X_{n+1}^\xi \rightarrow \mathbb{G}$ is a $2^{-(n+1)}$ -isometry and $\|u_{n+1,j}^\xi|_{X_n^\xi} - u_{n,j_n(\xi)}^\xi\| < 2 \cdot 2^{-n}$.

$\|u_{n,j_n(\xi)}^\xi\| < 2 \cdot 2^{-n}$. We already know that such a number exists. The function j_{n+1} is Borel, as

$$(j_{n+1})^{-1}(\{j\}) = \bigcup_{i=1}^{\infty} \left[(j_n)^{-1}(\{i\}) \cap \left(B_{n,i,j} \setminus \bigcup_{l=1}^{j-1} B_{n,i,l} \right) \right].$$

Our next step is to define an isometry $f_\xi : X_\xi \rightarrow \mathbb{G}$. For $x \in X_m^\xi$, the sequence $\{u_{n,j_n(\xi)}^\xi(x)\}_{n \geq m}$ is Cauchy, since $\|u_{n+1,j_{n+1}(\xi)}^\xi(x) - u_{n,j_n(\xi)}^\xi(x)\| \leq 2 \cdot 2^{-n} \|x\|_\xi$ for $n \geq m$ by (ii). Therefore, we can put

$$f_\xi(x) = \lim_{n \rightarrow \infty} u_{n,j_n(\xi)}^\xi(x), \quad x \in \bigcup_{m=1}^{\infty} X_m^\xi.$$

Using (i), we obtain

$$(1 + 2^{-n})^{-1} \|x\|_\xi \leq \|u_{n,j_n(\xi)}^\xi(x)\| \leq (1 + 2^{-n}) \|x\|_\xi, \quad x \in X_m^\xi, \quad n \geq m.$$

It follows that $\|f_\xi(x)\| = \|x\|_\xi$ for $x \in \bigcup_{m=1}^{\infty} X_m^\xi$ and so that there is a unique extension $f_\xi : X_\xi \rightarrow \mathbb{G}$ satisfying $\|f_\xi(x)\| = \|x\|_\xi$ for every $x \in X_\xi$.

Let us realize that the mapping

$$\chi_k : \Xi \rightarrow \mathbb{G}, \quad \chi_k(\xi) = f_\xi(x_k^\xi),$$

is Borel for every $k \in \mathbb{N}$. Since

$$\chi_k(\xi) = f_\xi(x_k^\xi) = \lim_{n \rightarrow \infty} u_{n,j_n(\xi)}^\xi(x_k^\xi) = \lim_{n \rightarrow \infty} u_{n,j_n(\xi)}^\xi(0, \dots, 0, \underset{k}{1}, 0, \dots, 0),$$

the mapping χ_k is the pointwise limit of a sequence of Borel mappings.

Finally, let us define the desired mapping \mathfrak{S} . We may suppose that \mathbb{G} is a subspace of $C([0, 1])$. This allows us to define

$$\mathfrak{S} : \Xi \rightarrow \mathcal{SE}(C([0, 1])), \quad \mathfrak{S}(\xi) = f_\xi(X_\xi), \quad \xi \in \Xi.$$

Since \mathfrak{S} fulfills the formula

$$\mathfrak{S}(\xi) = \overline{\text{span}} \{\chi_1(\xi), \chi_2(\xi), \dots\},$$

it is straightforward to show that it is a Borel mapping. \square

3. TSIRELSON TYPE SPACES

In this section, we will use Tsirelson type spaces introduced by S. A. Argyros and I. Deliyanni [1] to show that the class of spaces embeddable into c_0 is not Borel (Theorem 1.2). Those spaces are obtained by a generalization of the notion of an admissible family.

In fact, our approach is slightly different from the approach of Argyros and Deliyanni. The space defined below is derived from Tsirelson's original example \mathcal{T}_s^* , not from its dual \mathcal{T}_s (some comments on spaces derived from \mathcal{T}_s are provided in Remark 3.10). Moreover, we consider even more general systems of admissible families, including systems which lead to spaces quite different from \mathcal{T}_s^* (see Lemma 3.6). In spite of this, for our purposes, we use the symbol \mathcal{T}_s^* also for these non Tsirelson-like spaces.

Throughout this paper, we identify elements of $2^{\mathbb{N}}$ with subsets of \mathbb{N} . For this reason, members of $\mathcal{K}(2^{\mathbb{N}})$ represent systems of subsets of \mathbb{N} .

Let e_1, e_2, \dots be the canonical basis of c_{00} (i.e., $e_n = \mathbf{1}_{\{n\}}$). Let us recall that we denote $Ex = \mathbf{1}_E \cdot x$ for $E \subset \mathbb{N}$ and $x \in c_{00}$.

For $\mathcal{M} \in \mathcal{K}(2^{\mathbb{N}})$, a family $\{E_1, \dots, E_n\}$ of successive finite subsets of \mathbb{N} is said to be \mathcal{M} -admissible if an element of \mathcal{M} contains numbers m_1, \dots, m_n such that

$$m_1 \leq E_1 < m_2 \leq E_2 < \dots < m_n \leq E_n.$$

The system of all \mathcal{M} -admissible families is denoted by $\text{adm}(\mathcal{M})$.

Definition 3.1. For $\mathcal{M} \in \mathcal{K}(2^{\mathbb{N}})$, let $\Theta_{\mathcal{M}}$ be the smallest absolutely convex subset of c_{00} containing every $e_i, i \in \mathbb{N}$, and satisfying the property

$$\{E_1, \dots, E_n\} \in \text{adm}(\mathcal{M}) \ \& \ x_1, \dots, x_n \in \Theta_{\mathcal{M}} \quad \Rightarrow \quad \frac{1}{2} \sum_{k=1}^n E_k x_k \in \Theta_{\mathcal{M}}.$$

Let $\|\cdot\|_{\mathcal{M}}$ be the Minkowski gauge of $\Theta_{\mathcal{M}}$ and let $\mathcal{T}_s^*[\mathcal{M}, \frac{1}{2}]$ be a completion of $(c_{00}, \|\cdot\|_{\mathcal{M}})$.

First, we introduce without proof some simple facts about $\mathcal{T}_s^*[\mathcal{M}, \frac{1}{2}]$.

Fact 3.2. The sequence e_1, e_2, \dots is a 1-unconditional basis of $\mathcal{T}_s^*[\mathcal{M}, \frac{1}{2}]$.

Fact 3.3. If $\{E_1, \dots, E_n\}$ is \mathcal{M} -admissible and $x_1, \dots, x_n \in \mathcal{T}_s^*[\mathcal{M}, \frac{1}{2}]$, then

$$\left\| \sum_{k=1}^n E_k x_k \right\|_{\mathcal{M}} \leq 2 \sup_{1 \leq k \leq n} \|x_k\|_{\mathcal{M}}.$$

Fact 3.4. *If $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{K}(2^{\mathbb{N}})$ are two systems such that*

$$\{A \cap \{1, \dots, \ell\} : A \in \mathcal{M}_1\} = \{A \cap \{1, \dots, \ell\} : A \in \mathcal{M}_2\},$$

then $\|x\|_{\mathcal{M}_1} = \|x\|_{\mathcal{M}_2}$ for every $x \in \text{span}\{e_1, e_2, \dots, e_\ell\}$.

The following important property of $\mathcal{T}s^*[\mathcal{M}, \frac{1}{2}]$ follows from the proof of [1, Proposition 1.1].

Lemma 3.5 (Argyros, Deliyanni). *If \mathcal{M} consists of finite sets only, then e_1, e_2, \dots is a boundedly complete basis of $\mathcal{T}s^*[\mathcal{M}, \frac{1}{2}]$, and so $\mathcal{T}s^*[\mathcal{M}, \frac{1}{2}]$ can not be embedded isomorphically into c_0 .*

We are going to show that the space $\mathcal{T}s^*[\mathcal{M}, \frac{1}{2}]$ is considerably different from $\mathcal{T}s^*$ if \mathcal{M} contains an infinite set. This phenomenon is the heart of our argument.

Lemma 3.6. *If \mathcal{M} contains an infinite set, then $\mathcal{T}s^*[\mathcal{M}, \frac{1}{2}]$ is isomorphic to the c_0 -sum of a sequence of finite-dimensional spaces, and so $\mathcal{T}s^*[\mathcal{M}, \frac{1}{2}]$ can be embedded isomorphically into c_0 .*

Proof. Assuming that $\{m_1 < m_2 < \dots\} \in \mathcal{M}$, we show for every $x \in \mathcal{T}s^*[\mathcal{M}, \frac{1}{2}]$ that

$$\sup_{k \in \mathbb{N} \cup \{0\}} \|E_k x\|_{\mathcal{M}} \leq \|x\|_{\mathcal{M}} \leq \|E_0 x\|_{\mathcal{M}} + 2 \sup_{k \in \mathbb{N}} \|E_k x\|_{\mathcal{M}}$$

where $E_0 = \{1, \dots, m_1 - 1\}$ and $E_k = \{m_k, \dots, m_{k+1} - 1\}$. The first inequality follows from Fact 3.2. For $n \in \mathbb{N}$, the family $\{E_1, \dots, E_n\}$ is \mathcal{M} -admissible, and thus

$$\left\| \sum_{k=0}^n E_k x \right\|_{\mathcal{M}} \leq \|E_0 x\|_{\mathcal{M}} + \left\| \sum_{k=1}^n E_k x \right\|_{\mathcal{M}} \leq \|E_0 x\|_{\mathcal{M}} + 2 \sup_{1 \leq k \leq n} \|E_k x\|_{\mathcal{M}}$$

by Fact 3.3. Since $x = \sum_{k=0}^{\infty} E_k x$, the remaining inequality follows. \square

The following lemma, proof of which is essentially contained in [1], will be useful later.

Lemma 3.7. *If \mathcal{M} contains all three-element sets, then e_1, e_2, \dots is a shrinking basis of $\mathcal{T}s^*[\mathcal{M}, \frac{1}{2}]$. In particular, if \mathcal{M} contains all three-element sets but it consists of finite sets only, then $\mathcal{T}s^*[\mathcal{M}, \frac{1}{2}]$ is reflexive.*

Proof. Let e_1^*, e_2^*, \dots be the dual basic sequence. Given $x^* \in (\mathcal{T}s^*[\mathcal{M}, \frac{1}{2}])^*$, we want to show that

$$x^* = \sum_{i=1}^{\infty} x^*(e_i) e_i^*.$$

Assume the opposite, i.e., that

$$\varepsilon = \lim_{n \rightarrow \infty} \left\| x^* - \sum_{i=1}^{n-1} x^*(e_i) e_i^* \right\|_{\mathcal{M}} > 0$$

(we note that the sequence under the limit is non-increasing, due to 1-unconditionality). Choose $m_1 \in \mathbb{N}$ so that

$$\left\| x^* - \sum_{i=1}^{m_1-1} x^*(e_i) e_i^* \right\|_{\mathcal{M}} < \frac{4}{3} \varepsilon$$

and m_2, m_3, m_4 so that

$$\left\| \sum_{i=m_k}^{m_{k+1}-1} x^*(e_i) e_i^* \right\|_{\mathcal{M}} > \frac{8}{9} \varepsilon, \quad k = 1, 2, 3.$$

For $k = 1, 2, 3$, if we put $E_k = \{m_k, \dots, m_{k+1} - 1\}$, then we can find $x_k \in \mathcal{T}^s[\mathcal{M}, \frac{1}{2}]$, $\|x_k\|_{\mathcal{M}} \leq 1$, such that $x^*(E_k x_k) > \frac{8}{9} \varepsilon$. By our assumption, \mathcal{M} contains $\{m_1, m_2, m_3\}$, and so the family $\{E_1, E_2, E_3\}$ is \mathcal{M} -admissible. Hence, $\|E_1 x_1 + E_2 x_2 + E_3 x_3\|_{\mathcal{M}} \leq 2$ by Fact 3.3, and we obtain

$$3 \cdot \frac{8}{9} \varepsilon < x^*(E_1 x_1 + E_2 x_2 + E_3 x_3) < \frac{4}{3} \varepsilon \cdot \|E_1 x_1 + E_2 x_2 + E_3 x_3\|_{\mathcal{M}} \leq \frac{4}{3} \varepsilon \cdot 2,$$

a contradiction.

The second part of the statement follows from Lemma 3.5 and Theorem 2.3. \square

Lemma 3.8. *There exists a Borel mapping $\mathfrak{S} : \mathcal{K}(2^{\mathbb{N}}) \rightarrow \mathcal{SE}(C([0, 1]))$ such that $\mathfrak{S}(\mathcal{M})$ is isometric to $\mathcal{T}^s[\mathcal{M}, \frac{1}{2}]$ for every $\mathcal{M} \in \mathcal{K}(2^{\mathbb{N}})$.*

Proof. Due to Lemma 2.4, it is sufficient to realize that the function $\mathcal{M} \mapsto \|x\|_{\mathcal{M}}$ is Borel for every $x \in c_{00}$. We show that this function is continuous. By Fact 3.4, if ℓ is such that $x \in \text{span}\{e_1, \dots, e_{\ell}\}$, then the norm $\|x\|_{\mathcal{M}}$ depends only on $\{A \cap \{1, \dots, \ell\} : A \in \mathcal{M}\}$. For this reason, $\mathcal{K}(2^{\mathbb{N}})$ can be decomposed into finitely many clopen sets on which $\|x\|_{\mathcal{M}}$ is constant. \square

Proof of Theorem 1.2. It is easy to show that the class of all Banach spaces X which can be embedded isomorphically into c_0 (shortly $X \hookrightarrow c_0$) is analytic (see [3, Theorem 2.3]). Let us show that it is hard analytic. The set of all infinite subsets of \mathbb{N} is a G_{δ} but not F_{σ} subset of $2^{\mathbb{N}}$. By Theorem 2.2, the set

$$C = \{\mathcal{M} \in \mathcal{K}(2^{\mathbb{N}}) : \mathcal{M} \text{ contains an infinite set}\}$$

is complete analytic. Let \mathfrak{S} be a mapping provided by Lemma 3.8. Using Lemma 3.5 and Lemma 3.6, we obtain

$$\mathfrak{S}(\mathcal{M}) \hookrightarrow c_0 \quad \Leftrightarrow \quad \mathcal{M} \in \mathcal{C}.$$

It remains to apply Lemma 2.1. \square

Remark 3.9. (i) The space c_0 is not the only example for which the argument works. If a separable Banach space Z contains an isomorphic copy of c_0 but does not contain an infinite-dimensional reflexive subspace, then the class of all Banach spaces which can be embedded isomorphically into Z is complete analytic.

(ii) Let G_1, G_2, \dots be a dense sequence of finite-dimensional spaces (i.e., for every finite-dimensional Banach space G and every $\varepsilon > 0$, there is a bijective ε -isometry between G and some G_n). Then the class of all spaces isomorphic to $(\bigoplus G_n)_{c_0}$ is complete analytic. Indeed, the space $\mathcal{T}_s^*[\mathcal{M}, \frac{1}{2}] \oplus (\bigoplus G_n)_{c_0}$ is isomorphic to $(\bigoplus G_n)_{c_0}$ if and only if \mathcal{M} contains an infinite set.

Remark 3.10. Argyros and Deliyanni [1] defined a space $\mathcal{T}_s[\mathcal{M}, \frac{1}{2}]$ as the Banach space of sequences $x = \{x(i)\}_{i=1}^\infty$ with the basis $e_i = \mathbf{1}_{\{i\}}$ and with the implicitly defined norm

$$\|x\|_{\mathcal{M}} = \max \left\{ \|x\|_\infty, \frac{1}{2} \sup \left\{ \sum_{k=1}^n \|E_k x\|_{\mathcal{M}} : \{E_1, \dots, E_n\} \in \text{adm}(\mathcal{M}) \right\} \right\}.$$

It can be shown that the sequence e_i considered as a basis of $\mathcal{T}_s^*[\mathcal{M}, \frac{1}{2}]$ is dual to the same sequence e_i considered as a basis of $\mathcal{T}_s[\mathcal{M}, \frac{1}{2}]$. However, the duality between these two spaces is not warranted in our general setting.

Let us mention some notes and consequences of results from this section.

(i) If \mathcal{M} consists of finite sets only, then e_1, e_2, \dots is a shrinking basis of $\mathcal{T}_s[\mathcal{M}, \frac{1}{2}]$. In this case, $\mathcal{T}_s^*[\mathcal{M}, \frac{1}{2}]$ is the dual of $\mathcal{T}_s[\mathcal{M}, \frac{1}{2}]$.

(ii) If \mathcal{M} contains an infinite set, then $\mathcal{T}_s[\mathcal{M}, \frac{1}{2}]$ is isomorphic to the ℓ_1 -sum of a sequence of finite-dimensional spaces. In this case, the dual of $\mathcal{T}_s[\mathcal{M}, \frac{1}{2}]$ is not separable and contains $\mathcal{T}_s^*[\mathcal{M}, \frac{1}{2}]$ as a proper subspace. Notice also that $\mathcal{T}_s[\mathcal{M}, \frac{1}{2}]$ has the Schur property.

(iii) If \mathcal{M} contains all three-element sets, then e_1, e_2, \dots is a boundedly complete basis of $\mathcal{T}_s[\mathcal{M}, \frac{1}{2}]$, and $\mathcal{T}_s[\mathcal{M}, \frac{1}{2}]$ is the dual of $\mathcal{T}_s^*[\mathcal{M}, \frac{1}{2}]$.

(iv) If \mathcal{M} contains all three-element sets but it consists of finite sets only, then $\mathcal{T}_s[\mathcal{M}, \frac{1}{2}]$ is reflexive, as well as $\mathcal{T}_s^*[\mathcal{M}, \frac{1}{2}]$.

(v) There exists a Borel mapping $\mathfrak{S} : \mathcal{K}(2^{\mathbb{N}}) \rightarrow \mathcal{SE}(C([0, 1]))$ such that $\mathfrak{S}(\mathcal{M})$ is isometric to $\mathcal{Ts}[\mathcal{M}, \frac{1}{2}]$ for every $\mathcal{M} \in \mathcal{K}(2^{\mathbb{N}})$.

(vi) Let us denote by $\mathbb{N}^{[\leq 3]}$ the system of all subsets of \mathbb{N} with at most three elements. Then $\mathcal{M} \mapsto \mathfrak{S}(\mathcal{M} \cup \mathbb{N}^{[\leq 3]})$ is a Borel mapping which maps a complete analytic set into spaces with the Schur property and its complement into reflexive spaces. Therefore, it follows from the Tsirelson space method that the class of all separable Banach spaces with the Schur property is not coanalytic. Using a tree space method developed in [3], it can be shown that this class is not analytic (see [6, Theorem 27]). Our proof of the proper complexity result (Theorem 1.3) can be considered as a combination of these two methods.

(vii) The class of all spaces isomorphic to $(\bigoplus G_n)_{\ell_1}$ is complete analytic. Indeed, the space $\mathcal{Ts}[\mathcal{M}, \frac{1}{2}] \oplus (\bigoplus G_n)_{\ell_1}$ is isomorphic to $(\bigoplus G_n)_{\ell_1}$ if and only if \mathcal{M} contains an infinite set.

4. PRELIMINARIES II

By $\Lambda^{<\mathbb{N}}$ we denote the system of all finite sequences of elements of a set Λ , including the empty sequence \emptyset . That is,

$$\Lambda^{<\mathbb{N}} = \bigcup_{\ell=0}^{\infty} \Lambda^{\ell}$$

where $\Lambda^0 = \{\emptyset\}$. By $|\eta|$ we mean the length of $\eta \in \Lambda^{<\mathbb{N}}$. For $\sigma \in \Lambda^{\mathbb{N}}$, we denote by $\sigma|_{\ell}$ its initial segment $(\sigma(1), \dots, \sigma(\ell))$ of length $\ell \in \mathbb{N}$.

A subset T of $\Lambda^{<\mathbb{N}}$ is called a *tree on Λ* if it is downward closed, i.e.,

$$(\lambda_1, \lambda_2, \dots, \lambda_k) \in T \ \& \ j \leq k \quad \Rightarrow \quad (\lambda_1, \lambda_2, \dots, \lambda_j) \in T.$$

The set of all trees on Λ is denoted by $\text{Tr}(\Lambda)$ and endowed with the topology induced by the topology of $2^{\Lambda^{<\mathbb{N}}}$. The set of all infinite branches of $T \in \text{Tr}(\Lambda)$, i.e., sequences $\nu \in \Lambda^{\mathbb{N}}$ such that T contains all initial segments of ν , is denoted by $[T]$.

In what follows, we identify $(\Theta \times \Lambda)^{\ell}$ with $\Theta^{\ell} \times \Lambda^{\ell}$ and $(\Theta \times \Lambda)^{\mathbb{N}}$ with $\Theta^{\mathbb{N}} \times \Lambda^{\mathbb{N}}$. In this way, elements of a tree on $\Theta \times \Lambda$ are pairs of sequences of the same length and its infinite branches are elements of $\Theta^{\mathbb{N}} \times \Lambda^{\mathbb{N}}$.

If \mathcal{T} is a tree on $\Theta \times \Lambda$ and $\sigma \in \Theta^{\mathbb{N}}$, we define

$$\mathcal{T}(\sigma) \in \text{Tr}(\Lambda), \quad \mathcal{T}(\sigma) = \{\nu \in \Lambda^{<\mathbb{N}} : (\sigma|_{|\nu|}, \nu) \in \mathcal{T}\}.$$

We say that a tree T on \mathbb{N} is *ill-founded* ($T \in \text{IF}$) if it has an infinite branch (i.e., $[T] \neq \emptyset$). In the opposite case, we say that T is *well-founded* ($T \in \text{WF}$).

Lemma 4.1. *The set*

$$C = \{\mathcal{T} \in \text{Tr}(2 \times \mathbb{N}) : (\forall \sigma \in 2^{\mathbb{N}})(\mathcal{T}(\sigma) \in \text{IF})\}$$

is a Π_2^1 -complete subset of $\text{Tr}(2 \times \mathbb{N})$.

It is easy (and not necessary for our purposes actually) to check that C is a Π_2^1 set. To prove that it is Π_2^1 -hard, we will use the following well-known results:

- all uncountable standard Borel spaces are Borel isomorphic (see e.g. [18, (15.6)]) and, for this reason, it is possible to consider only $Y = \mathbb{N}^{\mathbb{N}}$ in the definition of a Γ -hard set (where $\Gamma = \Pi_2^1$) and only $Y = 2^{\mathbb{N}}$ in the definition of a Σ_2^1 set,
- a subset A of a Polish space X is analytic if and only if it is the projection of some closed $F \subset X \times \mathbb{N}^{\mathbb{N}}$ (see e.g. [18, (14.3)]),
- for a closed $F \subset \Lambda^{\mathbb{N}}$, we have $F = [T]$ for some $T \in \text{Tr}(\Lambda)$ (it is possible to collect all initial segments of elements of F , cf. [18, (2.4)]).

Proof. (cf. with [18, (37.11)]). Let A be a Π_2^1 subset of $\mathbb{N}^{\mathbb{N}}$. We need to find a Borel mapping $f : \mathbb{N}^{\mathbb{N}} \rightarrow \text{Tr}(2 \times \mathbb{N})$ such that $f^{-1}(C) = A$. There exists an analytic subset B of $\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that $\mathbb{N}^{\mathbb{N}} \setminus A = \text{proj}_{\mathbb{N}^{\mathbb{N}}}((\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus B)$, i.e.,

$$v \in A \iff \forall \sigma \in 2^{\mathbb{N}} : (v, \sigma) \in B$$

for all $v \in \mathbb{N}^{\mathbb{N}}$. There exists a closed subset F of $\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $B = \text{proj}_{\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}} F$, i.e.,

$$(v, \sigma) \in B \iff \exists \omega \in \mathbb{N}^{\mathbb{N}} : (v, \sigma, \omega) \in F.$$

There is a tree $\mathfrak{T} \in \text{Tr}(\mathbb{N} \times 2 \times \mathbb{N})$ such that $[\mathfrak{T}] = F$. We have

$$\begin{aligned} v \in A &\iff \forall \sigma \in 2^{\mathbb{N}} \exists \omega \in \mathbb{N}^{\mathbb{N}} : (v, \sigma, \omega) \in [\mathfrak{T}] \\ &\iff \forall \sigma \in 2^{\mathbb{N}} \exists \omega \in \mathbb{N}^{\mathbb{N}} : \omega \in [\mathfrak{T}(v, \sigma)] \\ &\iff \forall \sigma \in 2^{\mathbb{N}} : \mathfrak{T}(v, \sigma) \in \text{IF} \\ &\iff \mathfrak{T}(v) \in C \end{aligned}$$

for all $v \in \mathbb{N}^{\mathbb{N}}$, and it remains to note that the mapping $v \mapsto \mathfrak{T}(v)$ is continuous. \square

A bounded linear operator $T : X \rightarrow Y$ is called *weakly compact* if the image of the unit ball of X is relatively weakly compact in Y . The operator T is called *completely continuous* if it maps weakly convergent sequences to norm convergent ones.

We say that a Banach space X has the *Dunford-Pettis property* if every weakly compact operator $T : X \rightarrow Y$ from X into another Banach space Y is completely continuous.

In the remainder of this section, we prove the easy part of Theorem 1.3.

Lemma 4.2. *The class of all separable Banach spaces with the Dunford-Pettis property is Π_2^1 .*

During the proof, we will use the following known facts:

- an operator $T : X \rightarrow Y$ is completely continuous if and only if it maps weakly null sequences to null sequences,
- X has the Dunford-Pettis property if and only if every weakly compact operator $T : X \rightarrow c_0$ is completely continuous (see e.g. [7, Theorem 1]).

Proof. We prove first that $X \in \mathcal{SE}(C([0,1]))$ has the Dunford-Pettis property if and only if

$$\forall (x_1, x_2, \dots) \in B_{C[0,1]}^{\mathbb{N}} \quad \forall (y_1, y_2, \dots) \in B_{c_0}^{\mathbb{N}} : \text{(a) or (b) or (c) or (d) or (e),}$$

where we consider properties

- (a) $B_X \neq \overline{\{x_1, x_2, \dots\}}$,
- (b) $x_n \mapsto y_n$ does not define a bounded linear operator from $\overline{\text{span}\{x_1, x_2, \dots\}}$ into c_0 ,
- (c) $\{y_1, y_2, \dots\}$ is not relatively weakly compact,
- (d) x_2, x_4, x_6, \dots is not weakly null,
- (e) y_2, y_4, y_6, \dots is null.

Let us assume that X has the Dunford-Pettis property. For sequences x_1, x_2, \dots in $B_{C[0,1]}$ and y_1, y_2, \dots in B_{c_0} , we need to show that some of the five properties is satisfied. Let us suppose that none of properties (a), (b), (c), (d) is satisfied. So, $x_n \mapsto y_n$ defines a bounded linear operator $T : X \rightarrow c_0$ that is weakly compact. Moreover, the sequence x_2, x_4, x_6, \dots is weakly null. As X is assumed to have the Dunford-Pettis property, T is completely continuous, and thus it maps the weakly null sequence x_{2k} to a null sequence y_{2k} . It means that (e) is valid.

Let us assume that the formula is fulfilled for some $X \in \mathcal{SE}(C([0,1]))$. Let $T : X \rightarrow c_0$ be a weakly compact operator. We need to show that

T maps a weakly null sequence a_1, a_2, \dots to a null sequence. We may suppose that $\|T\| \leq 1$ and that $a_k \in B_X$. Let us put $x_{2k} = a_k$ and choose a sequence x_1, x_3, \dots that is dense in B_X . Moreover, let $y_n = Tx_n$. As T is weakly compact, the set $\{y_1, y_2, \dots\}$ is relatively weakly compact. So, none of properties (a), (b), (c), (d) is satisfied. Then (e) has to be valid. That is, the sequence $Ta_k = y_{2k}$ is null.

So, both implications are verified. To prove the lemma, it remains to show that each of the five properties define an analytic subset of $\mathcal{SE}(C([0, 1])) \times B_{C[0,1]}^{\mathbb{N}} \times B_{c_0}^{\mathbb{N}}$.

(a) The corresponding set is Borel. Indeed, if U is the open unit ball of $C[0, 1]$ and G_1, G_2, \dots is a basis of the norm topology of $C[0, 1]$, then $B_X = \overline{\{x_1, x_2, \dots\}}$ if and only if

$$\forall k \in \mathbb{N} : [X \cap U \cap G_k \neq \emptyset \Leftrightarrow \exists n \in \mathbb{N} : x_n \in G_k].$$

(b) The corresponding set is Borel. It is sufficient to realize that $x_n \mapsto y_n$ defines a bounded linear operator from $\overline{\text{span}}\{x_1, x_2, \dots\}$ into c_0 if and only if

$$\exists K \in \mathbb{N} \forall m \in \mathbb{N} \forall \alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{Q} : \left\| \sum_{n=1}^m \alpha_n y_n \right\| \leq K \left\| \sum_{n=1}^m \alpha_n x_n \right\|.$$

(c) Let us notice that B_{c_0} with the weak topology is a subspace of the topological product $[-1, 1]^{\mathbb{N}}$. A subset of B_{c_0} is relatively weakly compact if and only if its closure in $[-1, 1]^{\mathbb{N}}$ is still a subset of B_{c_0} . For this reason, $\{y_1, y_2, \dots\}$ is not relatively weakly compact if and only if

$$\begin{aligned} \exists z \in [-1, 1]^{\mathbb{N}} : & [\exists l \in \mathbb{N} \forall m \in \mathbb{N} \exists k \geq m : |z(k)| \geq 1/l] \\ & \& [\forall l, m \in \mathbb{N} \exists n \in \mathbb{N} \forall k \leq m : |y_n(k) - z(k)| < 1/l]. \end{aligned}$$

Hence, our set is a projection of a Borel subset of $\mathcal{SE}(C([0, 1])) \times B_{C[0,1]}^{\mathbb{N}} \times B_{c_0}^{\mathbb{N}} \times [-1, 1]^{\mathbb{N}}$.

(d) The corresponding set is analytic by [6, Theorem 20].

(e) It is easy to show that the corresponding set is Borel. □

5. TREE SPACES UPON TSIRELSON SPACES

In this section, we apply the construction of a tree space studied in [20] on Tsirelson type spaces presented above. This will enable us to show that some classes of Banach spaces have quite high complexity (Theorem 1.3).

For a finite sequence $\nu = (n_1, n_2, \dots, n_k) \in \mathbb{N}^{<\mathbb{N}}$, let $\tilde{\nu} = \{n_1 < n_1 + n_2 < \dots < \sum_{i=1}^k n_i\} \subset \mathbb{N}$. Similarly, for an infinite sequence $\nu = (n_1, n_2, \dots) \in \mathbb{N}^{\mathbb{N}}$, let $\tilde{\nu} = \{n_1 < n_1 + n_2 < \dots\} \subset \mathbb{N}$.

For every $T \in \text{Tr}(\mathbb{N})$, we define

$$\mathcal{M}_T = \{\tilde{\nu} : \nu \in T \cup [T] \text{ or } |\nu| \leq 3\}.$$

Let us note that \mathcal{M}_T belongs to $\mathcal{K}(2^{\mathbb{N}})$ and that it contains an infinite set if and only if T is ill-founded. Thus, we obtain from Lemma 3.6 and Lemma 3.7 a basic discovery about $\mathcal{T}s^*[\mathcal{M}_T, \frac{1}{2}]$.

Lemma 5.1. (1) *If $T \in \text{Tr}(\mathbb{N})$ is ill-founded, then $\mathcal{T}s^*[\mathcal{M}_T, \frac{1}{2}]$ is isomorphic to the c_0 -sum of a sequence of finite-dimensional spaces.*

(2) *If $T \in \text{Tr}(\mathbb{N})$ is well-founded, then $\mathcal{T}s^*[\mathcal{M}_T, \frac{1}{2}]$ is reflexive.*

The following observation follows from Fact 3.4.

Fact 5.2. *If T and S are two trees on \mathbb{N} which have the same sequences of length at most ℓ , then $\|x\|_{\mathcal{M}_T} = \|x\|_{\mathcal{M}_S}$ for every $x \in \text{span}\{e_1, e_2, \dots, e_\ell\}$.*

In particular, if $\mathcal{T} \in \text{Tr}(2 \times \mathbb{N})$ and $\sigma, \tau \in 2^{\mathbb{N}}$ satisfy $\sigma|_\ell = \tau|_\ell$, then $\|x\|_{\mathcal{M}_{\mathcal{T}(\sigma)}} = \|x\|_{\mathcal{M}_{\mathcal{T}(\tau)}}$ for every $x \in \text{span}\{e_1, e_2, \dots, e_\ell\}$.

Now, we are ready to introduce our tree space.

Definition 5.3. For $\mathcal{T} \in \text{Tr}(2 \times \mathbb{N})$, let $E_{\mathcal{T}}$ be defined as a completion of $c_{00}(2^{<\mathbb{N}} \setminus \{\emptyset\})$ with the norm

$$\|x\|_{\mathcal{T}} = \sup_{\sigma \in 2^{\mathbb{N}}} \left\| \sum_{\ell=1}^{\infty} x(\sigma|_\ell) e_\ell \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}}.$$

This space is defined according to [20, Definition 3.1]. Indeed, we can take $T = 2^{<\mathbb{N}} \setminus \{\emptyset\}$, $F_\sigma = \mathcal{T}s^*[\mathcal{M}_{\mathcal{T}(\sigma)}, \frac{1}{2}]$ and $f_\ell^\sigma = e_\ell$ for $\sigma \in 2^{\mathbb{N}}$ and $\ell \in \mathbb{N}$. The requirement from [20, Definition 3.1] is satisfied due to Fact 5.2.

Thus, results from [20, Section 3] are available. In particular, if we denote by $\{z_\eta : \eta \in 2^{<\mathbb{N}} \setminus \{\emptyset\}\}$ the canonical basis of $c_{00}(2^{<\mathbb{N}} \setminus \{\emptyset\})$, then this system forms a basis of $E_{\mathcal{T}}$.

The following two statements follow from [20, Fact 3.2], [20, Proposition 3.5] and Lemma 3.7.

Fact 5.4. *For every $\sigma \in 2^{\mathbb{N}}$, we have the 1-equivalence*

$$\left\| \sum_{\ell=1}^n \lambda_\ell z_{\sigma|_\ell} \right\|_{\mathcal{T}} = \left\| \sum_{\ell=1}^n \lambda_\ell e_\ell \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}}$$

of the sequences $z_{\sigma|_1}, z_{\sigma|_2}, \dots$ and e_1, e_2, \dots .

Therefore, $E_{\mathcal{T}}$ contains a 1-complemented copy of $\mathcal{I}\mathcal{S}^*[\mathcal{M}_{\mathcal{T}(\sigma)}, \frac{1}{2}]$. Consequently, $E_{\mathcal{T}}^*$ contains a 1-complemented copy of the dual of $\mathcal{I}\mathcal{S}^*[\mathcal{M}_{\mathcal{T}(\sigma)}, \frac{1}{2}]$.

Lemma 5.5. *The system $\{z_{\eta} : \eta \in 2^{<\mathbb{N}} \setminus \{\emptyset\}\}$ is a 1-unconditional shrinking basis of $E_{\mathcal{T}}$.*

The following crucial lemma will be proven in a separate section.

Lemma 5.6. *Let $\mathcal{T} \in \text{Tr}(2 \times \mathbb{N})$ be such that $\forall \sigma \in 2^{\mathbb{N}} : \mathcal{T}(\sigma) \in \text{IF}$. If x_1^*, x_2^*, \dots is a normalized sequence in $E_{\mathcal{T}}^*$ which converges to 0 in the w^* -topology, then it has a subsequence y_1^*, y_2^*, \dots such that*

$$\left\| \sum_{k=1}^n \lambda_k y_k^* \right\|_{\mathcal{T}} \geq \frac{1}{5} \sum_{k=1}^n |\lambda_k|$$

for all $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

It is not difficult to show that $E_{\mathcal{T}}^*$ has the Schur property if it satisfies the conclusion of Lemma 5.6. Nevertheless, we show that a bit more can be said.

For a bounded sequence x_1, x_2, \dots in a Banach space X , let us consider quantities

$$\text{ca}(x_n) = \inf_{m \in \mathbb{N}} \text{diam}\{x_n : n \geq m\},$$

$$\delta(x_n) = \sup_{\|x^*\| \leq 1} \inf_{m \in \mathbb{N}} \text{diam}\{x^*(x_n) : n \geq m\}.$$

Let $C \geq 1$. Following the authors of [15], we say that a Banach space X has the C -Schur property if

$$\text{ca}(x_n) \leq C\delta(x_n)$$

for any bounded sequence x_1, x_2, \dots in X .

Proposition 5.7. *Let $\mathcal{T} \in \text{Tr}(2 \times \mathbb{N})$.*

(1) *If $\forall \sigma \in 2^{\mathbb{N}} : \mathcal{T}(\sigma) \in \text{IF}$, then $E_{\mathcal{T}}^*$ has the 6-Schur property. Thus, $E_{\mathcal{T}}^*$ has the Schur property and the Dunford-Pettis property.*

(2) *In the opposite case, $E_{\mathcal{T}}^*$ contains a complemented infinite-dimensional reflexive subspace. Thus, $E_{\mathcal{T}}^*$ does not have the Schur property nor the Dunford-Pettis property.*

Proof. The part (2) follows immediately from Lemma 5.1(2) and Fact 5.4. Let us prove (1). Suppose that $\forall \sigma \in 2^{\mathbb{N}} : \mathcal{T}(\sigma) \in \text{IF}$ and that $\text{ca}(a_n^*) > 0$ for a bounded sequence a_1^*, a_2^*, \dots in $E_{\mathcal{T}}^*$. Let Q denote the non-empty set of all w^* -cluster points of a_n^* . To show that $\text{ca}(a_n^*) \leq 6\delta(a_n^*)$, we consider two possibilities.

Assume first that $\text{diam } Q \geq \frac{1}{6}\text{ca}(a_n^*)$. Let $\varepsilon > 0$ be given. There are $a^*, b^* \in Q$ with $\|b^* - a^*\|_{\mathcal{T}} > \frac{1}{6}\text{ca}(a_n^*) - \varepsilon$. Further, there is $x \in E_{\mathcal{T}}$, $\|x\|_{\mathcal{T}} \leq 1$, such that $(b^* - a^*)(x) > \frac{1}{6}\text{ca}(a_n^*) - \varepsilon$. For every $m \in \mathbb{N}$, we can find $k, l \geq m$ such that $a_k^*(x) < a^*(x) + \varepsilon$ and $a_l^*(x) > b^*(x) - \varepsilon$. Then $\text{diam}\{a_n^*(x) : n \geq m\} \geq a_l^*(x) - a_k^*(x) > b^*(x) - a^*(x) - 2\varepsilon > \frac{1}{6}\text{ca}(a_n^*) - 3\varepsilon$. It follows that $\delta(a_n^*) \geq \frac{1}{6}\text{ca}(a_n^*) - 3\varepsilon$. Since the argument works for any $\varepsilon > 0$, we obtain $\delta(a_n^*) \geq \frac{1}{6}\text{ca}(a_n^*)$.

Now, assume that $\text{diam } Q < \frac{1}{6}\text{ca}(a_n^*)$. Notice that $\text{dist}(a_n^*, Q) \geq \frac{5}{12}\text{ca}(a_n^*)$ for infinitely many n 's. Therefore, we can find a subsequence $a_{n_k}^*$ such that $\text{dist}(a_{n_k}^*, Q) \geq \frac{5}{12}\text{ca}(a_n^*)$ for every k . We may suppose that $a_{n_k}^*$ converges to some $a^* \in Q$ in the w^* -topology. Let us put

$$x_k^* = \frac{1}{\|a_{n_k}^* - a^*\|_{\mathcal{T}}} (a_{n_k}^* - a^*), \quad k = 1, 2, \dots$$

By Lemma 5.6, there is a subsequence $x_{k_l}^*$ such that

$$\left\| \sum_{l=1}^m \lambda_l x_{k_l}^* \right\|_{\mathcal{T}} \geq \frac{1}{5} \sum_{l=1}^m |\lambda_l|$$

for all $m \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Using the Hahn-Banach extension theorem, we can find $x^{**} \in E_{\mathcal{T}}^{**}$ with $\|x^{**}\|_{\mathcal{T}} \leq 1$ such that

$$x^{**}(x_{k_l}^*) = \frac{(-1)^l}{5}, \quad l = 1, 2, \dots$$

Then

$$\begin{aligned} (-1)^l x^{**}(a_{n_{k_l}}^* - a^*) &= (-1)^l \|a_{n_{k_l}}^* - a^*\|_{\mathcal{T}} x^{**}(x_{k_l}^*) \\ &= \frac{1}{5} \|a_{n_{k_l}}^* - a^*\|_{\mathcal{T}} \geq \frac{1}{5} \cdot \frac{5}{12} \text{ca}(a_n^*) = \frac{1}{12} \text{ca}(a_n^*), \end{aligned}$$

and so

$$x^{**}(a_{n_{k_{2j}}}^*) \geq x^{**}(a^*) + \frac{1}{12} \text{ca}(a_n^*), \quad x^{**}(a_{n_{k_{2j+1}}}^*) \leq x^{**}(a^*) - \frac{1}{12} \text{ca}(a_n^*).$$

It follows that $\delta(a_n^*) \geq \frac{1}{6}\text{ca}(a_n^*)$. \square

Lemma 5.8. *There exist Borel mappings $\mathfrak{S}, \mathfrak{S}^* : \text{Tr}(2 \times \mathbb{N}) \rightarrow \mathcal{SE}(C([0, 1]))$ such that $\mathfrak{S}(\mathcal{T})$ is isometric to $E_{\mathcal{T}}$ and $\mathfrak{S}^*(\mathcal{T})$ is isometric to $E_{\mathcal{T}}^*$ for every $\mathcal{T} \in \text{Tr}(2 \times \mathbb{N})$.*

Proof. Let us prove the existence of \mathfrak{S} first. Due to Lemma 2.4, it is sufficient to show that the function $\mathcal{T} \mapsto \|x\|_{\mathcal{T}}$ is Borel for every

$x \in c_{00}(2^{<\mathbb{N}} \setminus \{\emptyset\})$. Let Σ be a finite subset of $2^{\mathbb{N}}$ such that x is supported by initial segments of elements of Σ . By Fact 5.2, we have

$$\|x\|_{\mathcal{T}} = \max_{\sigma \in \Sigma} \left\| \sum_{\ell=1}^{\infty} x(\sigma|_{\ell}) e_{\ell} \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}}, \quad \mathcal{T} \in \text{Tr}(2 \times \mathbb{N}).$$

For this reason, $\mathcal{T} \mapsto \|x\|_{\mathcal{T}}$ is the maximum of finitely many continuous functions. Indeed,

- the mapping $\mathcal{T} \mapsto \mathcal{T}(\sigma)$ is continuous from $\text{Tr}(2 \times \mathbb{N})$ into $\text{Tr}(\mathbb{N})$ for every $\sigma \in \Sigma$ (easy),
- the mapping $T \mapsto \mathcal{M}_T$ is continuous from $\text{Tr}(\mathbb{N})$ into $\mathcal{K}(2^{\mathbb{N}})$ (just apply the fact that the Vietoris topology on $\mathcal{K}(2^{\mathbb{N}})$ is generated by the sets $\{\mathcal{M} \in \mathcal{K}(2^{\mathbb{N}}) : \mathcal{M} \cap \Delta_{\eta} \neq \emptyset\}$ and their complements, where η varies over sequences from $2^{<\mathbb{N}}$ and Δ_{η} denotes the clopen set $\{\sigma \in 2^{\mathbb{N}} : \sigma|_{|\eta|} = \eta\}$),
- the function $\mathcal{M} \mapsto \|y\|_{\mathcal{M}}$ is continuous for every $y \in c_{00}$ (see the proof of Lemma 3.8).

Now, let us prove the existence of \mathfrak{S}^* . By Lemma 5.5, the system $\{z_{\eta} : \eta \in 2^{<\mathbb{N}} \setminus \{\emptyset\}\}$ is a shrinking basis of $E_{\mathcal{T}}$ for every \mathcal{T} . Using Lemma 2.4 again, it is therefore sufficient to show that the function $\mathcal{T} \mapsto \|x^*\|_{\mathcal{T}}$ is Borel for every linear form x^* on $c_{00}(2^{<\mathbb{N}} \setminus \{\emptyset\})$ with a finite support. Let $S \subset c_{00}(2^{<\mathbb{N}} \setminus \{\emptyset\})$ be the countable set of all non-zero vectors with rational coordinates. Then

$$\|x^*\|_{\mathcal{T}} = \sup_{x \in S} \frac{x^*(x)}{\|x\|_{\mathcal{T}}}, \quad \mathcal{T} \in \text{Tr}(2 \times \mathbb{N}).$$

It follows from the first part of the proof that $\mathcal{T} \mapsto \|x^*\|_{\mathcal{T}}$ is Borel. \square

Proof of Theorem 1.3. The class of all separable Banach spaces with the Schur property is Π_2^1 ([6, Theorem 28]), as well as the class of all separable Banach spaces with the Dunford-Pettis property (Lemma 4.2). Let us show that both classes are Π_2^1 -hard. Let \mathfrak{S}^* be a mapping provided by Lemma 5.8. Using Proposition 5.7, we obtain

$$\begin{aligned} \forall \sigma \in 2^{\mathbb{N}} : \mathcal{T}(\sigma) \in \text{IF} &\Leftrightarrow \mathfrak{S}^*(\mathcal{T}) \text{ has the Schur property} \\ &\Leftrightarrow \mathfrak{S}^*(\mathcal{T}) \text{ has the Dunford-Pettis property} \end{aligned}$$

for every $\mathcal{T} \in \text{Tr}(2 \times \mathbb{N})$. It means that $\mathfrak{S}^*(\mathcal{T})$ has the Schur property (Dunford-Pettis property) if and only if $\mathcal{T} \in C$, where $C \subset \text{Tr}(2 \times \mathbb{N})$ is the Π_2^1 -complete set from Lemma 4.1. It remains to apply Lemma 2.1. \square

Remark 5.9. (i) It is possible to use [2, Theorem 3.2] to quantify the Schur property of $E_{\mathcal{T}}^*$ in a bit different direction. If the assumption $\forall \sigma \in 2^{\mathbb{N}} : \mathcal{T}(\sigma) \in \text{IF}$ is met, then $E_{\mathcal{T}}^*$ has the strong Schur property in the sense that every bounded sequence a_1^*, a_2^*, \dots in $E_{\mathcal{T}}^*$ with $\inf\{\|a_n^* - a_m^*\|_{\mathcal{T}} : n \neq m\} > \varepsilon$ has a subsequence $a_{n_j}^*$ such that

$$\left\| \sum_{j=1}^k \lambda_j a_{n_j}^* \right\|_{\mathcal{T}} \geq \frac{\varepsilon}{12} \sum_{j=1}^k |\lambda_j|$$

for all $k \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}$.

(ii) If the assumption $\forall \sigma \in 2^{\mathbb{N}} : \mathcal{T}(\sigma) \in \text{IF}$ is met, a quantitative version of the Dunford-Pettis property of $E_{\mathcal{T}}^*$ and of $E_{\mathcal{T}}$ can be obtained as well (see [14, Proposition 6.4 and Theorem 6.5]).

(iii) In [20], a question was considered whether the proposed tree space method can be used for amalgamating of spaces with the Schur property (see [20, Remark 3.7(c)] for the exact formulation). Proposition 5.7(1) shows a concrete example of a family of spaces with the Schur property for which the answer is positive, although not trivial. Let us note that to prove that $E_{\mathcal{T}}^*$ has the Schur property would be much simpler if we had the positive answer to the following question: *Does a Banach space X has necessarily the Schur property if it has a subset W such that $\overline{\text{co}} W = B_X$ and every weakly convergent sequence of elements of W is convergent in the norm?*

6. PROOF OF LEMMA 5.6

Let $\mathcal{T} \in \text{Tr}(2 \times \mathbb{N})$ satisfying $\forall \sigma \in 2^{\mathbb{N}} : \mathcal{T}(\sigma) \in \text{IF}$ be given, together with a normalized sequence x_1^*, x_2^*, \dots in $E_{\mathcal{T}}^*$ converging to 0 in the w^* -topology. Let us recall that our task is to find a subsequence y_1^*, y_2^*, \dots such that

$$\left\| \sum_{k=1}^n \lambda_k y_k^* \right\|_{\mathcal{T}} \geq \frac{1}{5} \sum_{k=1}^n |\lambda_k|$$

for all $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Note that each $x^* \in E_{\mathcal{T}}^*$ can be viewed as the system $\{x^*(z_{\eta})\}_{\eta \in 2^{<\mathbb{N}} \setminus \{\emptyset\}}$ of real numbers. By Lemma 5.5, elements with a finite support are dense in $E_{\mathcal{T}}^*$. Note also that $x_k^*(z_{\eta}) \rightarrow 0$ for every η .

By the passage to a subsequence and a small perturbation, we can obtain a sequence (which is denoted also x_k^*) satisfying:

(WLOG-1) There are $1 \leq p_1 \leq q_1 < p_2 \leq q_2 < \dots$ such that x_k^* is supported by sequences of length in $[p_k, q_k]$. (Because of the

perturbation, we just need to prove the desired inequality with a constant better than $\frac{1}{5}$).

For every k , let $x_k \in E_{\mathcal{T}}$ be such that $x_k^*(x_k) = \|x_k\|_{\mathcal{T}} = 1$ and x_k is supported by sequences of length in $[p_k, q_k]$ (as well as x_k^*). Let

$$x_k = \sum_{\eta \in 2^{p_k}} x_{k,\eta}$$

be the decomposition of x_k such that $x_{k,\eta}$ is supported by sequences which extend η .

Let us denote $\Delta = 2^{\mathbb{N}}$ and $\Delta_{\eta} = \{\sigma \in \Delta : \sigma|_{|\eta|} = \eta\}$. Let Σ_{ℓ} be the σ -algebra generated by the sets $\Delta_{\eta}, \eta \in 2^{\ell}$. The formula

$$\mathfrak{m}_k(\Delta_{\eta}) = x_k^*(x_{k,\eta}), \quad \eta \in 2^{p_k},$$

defines a probability measure on Σ_{p_k} . (We have $x_k^*(x_{k,\eta}) \geq 0$ because $1 - x_k^*(x_{k,\eta}) = x_k^*(x_k - x_{k,\eta}) \leq \|x_k - x_{k,\eta}\|_{\mathcal{T}} \leq \|x_k\|_{\mathcal{T}} = 1$).

Every \mathfrak{m}_k can be extended to a Borel probability measure on Δ . The sequence of these extensions has a cluster point in the w^* -topology of $C(\Delta)^*$. We can therefore assume that:

(WLOG-2) The measures \mathfrak{m}_k converge to a Borel probability measure \mathfrak{m} on Δ in the sense that $\mathfrak{m}_k(\Delta_{\eta}) \rightarrow \mathfrak{m}(\Delta_{\eta})$ for every $\eta \in 2^{<\mathbb{N}}$.

Claim 6.1. *There is an increasing sequence $s_1 < s_2 < \dots$ of natural numbers and a closed subset $\Gamma \subset \Delta$ such that $\mathfrak{m}(\Gamma) \geq \frac{7}{8}$ and, for every $\sigma \in \Gamma$, the system $\mathcal{M}_{\mathcal{T}(\sigma)}$ contains a set which intersects $[s_n, s_{n+1})$ for each $n \in \mathbb{N}$.*

Proof. The set $[\mathcal{T}]$ of all infinite branches of \mathcal{T} is a closed subset of $2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ whose sections $[\mathcal{T}(\sigma)], \sigma \in 2^{\mathbb{N}}$, are non-empty due to the assumption of the lemma. By the Jankov-von Neumann uniformization theorem (see e.g. (18.1) in [18]), there exists a selector $\sigma \in 2^{\mathbb{N}} \mapsto v_{\sigma} \in [\mathcal{T}(\sigma)]$ which is measurable with respect to the σ -algebra generated by the analytic subsets of $2^{\mathbb{N}}$. By a theorem of Lusin (see e.g. (21.10) in [18]), members of this σ -algebra are $\overline{\mathfrak{m}}$ -measurable, where $\overline{\mathfrak{m}}$ denotes the completion of \mathfrak{m} .

For natural numbers $r \leq s$, let us denote

$$\Lambda_{r,s} = \{\sigma \in \Delta : [r, s) \cap v_{\sigma} \neq \emptyset\}.$$

For every $r \in \mathbb{N}$ and $\varepsilon > 0$, since $\bigcup_{s=r}^{\infty} \Lambda_{r,s} = \Delta$, there is $s \geq r$ such that $\overline{\mathfrak{m}}(\Lambda_{r,s}) \geq 1 - \varepsilon$.

Let us take a sequence $\varepsilon_1, \varepsilon_2, \dots$ of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{8}$. Let $s_1 = 1$ and let s_2, s_3, \dots be chosen in the way that

$$\overline{\mathfrak{m}}(\Lambda_{s_n, s_{n+1}}) \geq 1 - \varepsilon_n$$

for $n = 1, 2, \dots$. The set

$$\Gamma_0 = \bigcap_{n=1}^{\infty} \Lambda_{s_n, s_{n+1}}$$

fulfills $\overline{m}(\Gamma_0) > 1 - \frac{1}{8} = \frac{7}{8}$ and, for every $\sigma \in \Gamma_0$, the system $\mathcal{M}_{\mathcal{T}(\sigma)}$ contains the set \tilde{v}_σ which intersects $[s_n, s_{n+1})$ for each $n \in \mathbb{N}$. Finally, let $\Gamma \subset \Gamma_0$ be a compact subset with $\overline{m}(\Gamma) \geq \frac{7}{8}$. \square

Now, let us consider such $s_1 < s_2 < \dots$ and $\Gamma \subset \Delta$ as in Claim 6.1. Let $\theta \subset 2^{<\mathbb{N}} \setminus \{\emptyset\}$ denote the set of all non-empty initial segments of sequences from Γ . Let

$$x_k = u_k + v_k$$

be the decomposition of x_k such that u_k is supported by θ and v_k is supported by the complement of θ . Let

$$x_{k,\eta} = u_{k,\eta} + v_{k,\eta}$$

be the analogous decomposition of $x_{k,\eta}$.

We are ready to establish our third and last additional assumption.

(WLOG-3) One of the following possibilities takes place:

- (I) $x_k^*(u_k) \geq \frac{1}{2}$ for every k .
- (II) $x_k^*(v_k) > \frac{1}{2}$ for every k .

Claim 6.2. *There is a subsequence y_j^* of x_k^* such that, for every $m \in \mathbb{N}$, there is $w \in E_{\mathcal{T}}$ with $\|w\|_{\mathcal{T}} \leq 1$ and*

$$y_j^*(w) \geq \frac{1}{4}, \quad j = 1, 2, \dots, m.$$

Before the proof of this claim, we show that the provided subsequence y_j^* of x_k^* has the desired property. Given $m \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, taking a suitable w and using that x_1^*, x_2^*, \dots have disjoint supports, we obtain

$$\begin{aligned} \left\| \sum_{j=1}^m \lambda_j y_j^* \right\|_{\mathcal{T}} &= \left\| \sum_{j=1}^m |\lambda_j| y_j^* \right\|_{\mathcal{T}} \geq \left\| \sum_{j=1}^m |\lambda_j| y_j^* \right\|_{\mathcal{T}} \|w\|_{\mathcal{T}} \\ &\geq \left(\sum_{j=1}^m |\lambda_j| y_j^* \right)(w) = \sum_{j=1}^m |\lambda_j| y_j^*(w) \geq \frac{1}{4} \sum_{j=1}^m |\lambda_j|. \end{aligned}$$

Let us recall that a better constant than $\frac{1}{5}$ is needed because of the perturbation done at the beginning of this section. As the constant $\frac{1}{4}$ is greater than $\frac{1}{5}$, Lemma 5.6 is proven.

It remains to prove Claim 6.2. We consider separately the possibilities (I) and (II) introduced above.

Proof of Claim 6.2, case (I). We choose a subsequence $y_j^* = x_{k_j}^*$ in the way that

$$s_{n_1} < s_{n_1+1} < p_{k_1} \leq q_{k_1} < s_{n_2} < s_{n_2+1} < p_{k_2} \leq q_{k_2} < s_{n_3} < s_{n_3+1} < \dots$$

for some suitable n_1, n_2, \dots . Let us consider the intervals in \mathbb{N} given by

$$I_j = [p_{k_j}, q_{k_j}], \quad j = 1, 2, \dots$$

Due to the choice of $s_1 < s_2 < \dots$ and $\Gamma \subset \Delta$ (see Claim 6.1), the family $\{I_1, \dots, I_m\}$ is $\mathcal{M}_{\mathcal{T}(\sigma)}$ -admissible for every $m \in \mathbb{N}$ and every $\sigma \in \Gamma$.

Given $m \in \mathbb{N}$, let us define

$$w = \frac{1}{2} \sum_{i=1}^m u_{k_i}.$$

Using (I), we obtain for $1 \leq j \leq m$ that

$$y_j^*(w) = x_{k_j}^*(w) = \frac{1}{2} \sum_{i=1}^m x_{k_j}^*(u_{k_i}) = \frac{1}{2} x_{k_j}^*(u_{k_j}) \geq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},$$

so it is sufficient to verify that $\|w\|_{\mathcal{T}} \leq 1$, i.e., that

$$\forall \sigma \in 2^{\mathbb{N}} : \quad \left\| \sum_{\ell=1}^{\infty} w(\sigma|_{\ell}) e_{\ell} \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}} \leq 1.$$

Consider $\sigma \in \Gamma$ first. For $1 \leq i \leq m$, the point $\sum_{\ell=1}^{\infty} u_{k_i}(\sigma|_{\ell}) e_{\ell}$ is supported by $[p_{k_i}, q_{k_i}] = I_i$. Using $\mathcal{M}_{\mathcal{T}(\sigma)}$ -admissibility of $\{I_1, \dots, I_m\}$ and Fact 3.3, we obtain

$$\begin{aligned} \left\| \sum_{\ell=1}^{\infty} w(\sigma|_{\ell}) e_{\ell} \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}} &= \frac{1}{2} \left\| \sum_{i=1}^m \sum_{\ell=1}^{\infty} u_{k_i}(\sigma|_{\ell}) e_{\ell} \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}} \\ &\leq \sup_{1 \leq i \leq m} \left\| \sum_{\ell=1}^{\infty} u_{k_i}(\sigma|_{\ell}) e_{\ell} \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}} \\ &\leq \sup_{1 \leq i \leq m} \|u_{k_i}\|_{\mathcal{T}} \leq \sup_{1 \leq i \leq m} \|x_{k_i}\|_{\mathcal{T}} = 1. \end{aligned}$$

Now, consider $\sigma \in \Delta \setminus \Gamma$. Note that w is supported by θ and that $\sigma \notin \Gamma = \bar{\Gamma} = [\theta \cup \{\emptyset\}]$. Let η be the longest initial segment of σ

belonging to $\theta \cup \{\emptyset\}$. Then η is an initial segment of some $\sigma' \in \Gamma$. Due to Fact 5.2, we obtain

$$\begin{aligned} \left\| \sum_{\ell=1}^{\infty} w(\sigma|_{\ell}) e_{\ell} \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}} &= \left\| \sum_{\ell=1}^{|\eta|} w(\sigma|_{\ell}) e_{\ell} \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}} \\ &= \left\| \sum_{\ell=1}^{|\eta|} w(\sigma'|_{\ell}) e_{\ell} \right\|_{\mathcal{M}_{\mathcal{T}(\sigma')}} \leq \left\| \sum_{\ell=1}^{\infty} w(\sigma'|_{\ell}) e_{\ell} \right\|_{\mathcal{M}_{\mathcal{T}(\sigma')}} \leq 1. \end{aligned}$$

This completes the verification of $\|w\|_{\mathcal{T}} \leq 1$. \square

Proof of Claim 6.2, case (II). Let $\Gamma^{(\ell)}$ denote the smallest set in Σ_{ℓ} containing Γ , that is $\Gamma^{(\ell)} = \{\sigma \in \Delta : \sigma|_{\ell} \in \theta\}$. We choose a subsequence $y_j^* = x_{k_j}^*$ in the way that

$$|\mathbf{m}_{k_{j+1}}(\Gamma^{(q_{k_j})}) - \mathbf{m}(\Gamma^{(q_{k_j})})| \leq \frac{1}{8}, \quad j = 1, 2, \dots$$

We have

$$\mathbf{m}_{k_{j+1}}(\Gamma^{(q_{k_j})}) \geq \mathbf{m}(\Gamma^{(q_{k_j})}) - \frac{1}{8} \geq \mathbf{m}(\Gamma) - \frac{1}{8} \geq \frac{7}{8} - \frac{1}{8} = \frac{3}{4}.$$

Let us define $w_{k_1} = v_{k_1}$ and

$$w_{k_{j+1}} = v_{k_{j+1}} - \sum_{\eta \in 2^{p_{k_{j+1}}}, \Delta_{\eta} \cap \Gamma^{(q_{k_j})} = \emptyset} v_{k_{j+1}, \eta}, \quad j = 1, 2, \dots$$

Then, using (II), we obtain $x_{k_1}^*(w_{k_1}) = x_{k_1}^*(v_{k_1}) \geq \frac{1}{2} \geq \frac{1}{4}$ and

$$\begin{aligned} x_{k_{j+1}}^*(w_{k_{j+1}}) &= x_{k_{j+1}}^*(v_{k_{j+1}}) - \sum_{\eta \in 2^{p_{k_{j+1}}}, \Delta_{\eta} \cap \Gamma^{(q_{k_j})} = \emptyset} x_{k_{j+1}}^*(v_{k_{j+1}, \eta}) \\ &\geq \frac{1}{2} - \sum_{\eta \in 2^{p_{k_{j+1}}}, \Delta_{\eta} \cap \Gamma^{(q_{k_j})} = \emptyset} x_{k_{j+1}}^*(x_{k_{j+1}, \eta}) \\ &= \frac{1}{2} - \sum_{\eta \in 2^{p_{k_{j+1}}}, \Delta_{\eta} \cap \Gamma^{(q_{k_j})} = \emptyset} \mathbf{m}_{k_{j+1}}(\Delta_{\eta}) \\ &= \frac{1}{2} - \mathbf{m}_{k_{j+1}}(\Delta \setminus \Gamma^{(q_{k_j})}) \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}. \end{aligned}$$

(We have $x_{k_{j+1}}^*(v_{k_{j+1}, \eta}) \leq x_{k_{j+1}}^*(x_{k_{j+1}, \eta})$ because $1 - x_{k_{j+1}}^*(u_{k_{j+1}, \eta}) = x_{k_{j+1}}^*(x_{k_{j+1}} - u_{k_{j+1}, \eta}) \leq \|x_{k_{j+1}} - u_{k_{j+1}, \eta}\|_{\mathcal{T}} \leq \|x_{k_{j+1}}\|_{\mathcal{T}} = 1$).

We claim that every infinite branch intersects the support of at most one w_{k_j} . Let us make two observations first.

(a) If $\sigma \in \Gamma^{(q_{k_i})}$, then the branch $\{\sigma|_1, \sigma|_2, \dots\}$ does not intersect the support of w_{k_j} for $j \leq i$.

Indeed, the initial segments $\sigma|_1, \sigma|_2, \dots, \sigma|_{q_{k_i}}$ belong to θ . In particular, the initial segments $\sigma|_{p_{k_j}}, \sigma|_{p_{k_j}+1}, \dots, \sigma|_{q_{k_j}}$ belong to θ . The support of w_{k_j} is disjoint from θ , and so $\{\sigma|_1, \sigma|_2, \dots\}$ does not intersect the support of w_{k_j} .

(b) If $\sigma \notin \Gamma^{(q_{k_i})}$, then the branch $\{\sigma|_1, \sigma|_2, \dots\}$ does not intersect the support of $w_{k_{j+1}}$ for $j \geq i$.

Indeed, we have

$$\Delta_{\sigma|_{q_{k_i}}} \cap \Gamma^{(q_{k_i})} = \emptyset$$

and, in particular,

$$\Delta_{\sigma|_{p_{k_{j+1}}}} \cap \Gamma^{(q_{k_j})} = \emptyset.$$

The sequence $\eta = \sigma|_{p_{k_{j+1}}}$ appears in the sum in the definition of $w_{k_{j+1}}$, and so $w_{k_{j+1}}(\sigma|_\ell) = 0$ for every $\ell \geq p_{k_{j+1}}$.

Now, we obtain from (a) and (b) that

- if $\sigma \in \Delta \setminus \Gamma^{(q_{k_1})}$, then the branch $\{\sigma|_1, \sigma|_2, \dots\}$ does not intersect the support of w_{k_j} for $j \neq 1$,
- if $\sigma \in \Gamma^{(q_{k_i})} \setminus \Gamma^{(q_{k_{i+1}})}$ for some i , then the branch $\{\sigma|_1, \sigma|_2, \dots\}$ does not intersect the support of w_{k_j} for $j \neq i+1$,
- if $\sigma \in \bigcap_{i=1}^{\infty} \Gamma^{(q_{k_i})}$, then the branch $\{\sigma|_1, \sigma|_2, \dots\}$ does not intersect the support of w_{k_j} for every j .

So, we have shown that every infinite branch intersects the support of at most one w_{k_j} . Now, given $m \in \mathbb{N}$, let us define

$$w = \sum_{i=1}^m w_{k_i}.$$

For every $\sigma \in 2^{\mathbb{N}}$, there is $j \leq m$ such that $w(\sigma|_\ell) = w_{k_j}(\sigma|_\ell)$ for each $\ell \in \mathbb{N}$ (we can choose any $j \leq m$ if $w(\sigma|_\ell) = 0$ for each ℓ), and so

$$\left\| \sum_{\ell=1}^{\infty} w(\sigma|_\ell) e_\ell \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}} = \left\| \sum_{\ell=1}^{\infty} w_{k_j}(\sigma|_\ell) e_\ell \right\|_{\mathcal{M}_{\mathcal{T}(\sigma)}} \leq \|w_{k_j}\|_{\mathcal{T}} \leq \|x_{k_j}\|_{\mathcal{T}} = 1.$$

It follows that $\|w\|_{\mathcal{T}} \leq 1$. At the same time, for $j \leq m$, we have

$$y_j^*(w) = x_{k_j}^*(w) = \sum_{i=1}^m x_{k_j}^*(w_{k_i}) = x_{k_j}^*(w_{k_j}) \geq \frac{1}{4},$$

and thus w works. □

7. A QUESTION

The aim of this short final section is a discussion on the complexity of the isomorphism class of c_0 (see Question 1.1) and the formulation of a related problem concerning equivalent norms on c_0 . First, let us mention a remarkable conjecture from [12].

Conjecture 7.1 (Godefroy, Kalton, Lancien). *If X is a Banach space with summable Szlenk index whose dual X^* is isomorphic to ℓ_1 , then X is isomorphic to c_0 .*

The validity of this conjecture would imply that every Banach space uniformly isomorphic to c_0 was actually isomorphic to c_0 . As noted by G. Godefroy [10], confirming the conjecture would give also the positive answer to Question 1.1. We are going to provide a variant of this approach.

We start with an investigation of subspaces of c_0 based on a result of N. J. Kalton [16]. Let us recall that we denote

$$\text{ca}(x_n) = \inf_{m \in \mathbb{N}} \text{diam}\{x_n : n \geq m\}$$

for a bounded sequence x_1, x_2, \dots in a Banach space X .

Theorem 7.2. *For a separable Banach space X , the following assertions are equivalent:*

- (i) X can be embedded isomorphically into c_0 .
- (ii) There exists a bounded function $\mu : B_{X^*} \rightarrow \mathbb{R}$ such that

$$\liminf_{n \rightarrow \infty} \mu(x^* + x_n^*) \geq \mu(x^*) + \liminf_{n \rightarrow \infty} \|x_n^*\|$$

whenever $x^* \in B_{X^*}$, $x^* + x_n^* \in B_{X^*}$ and x_1^*, x_2^*, \dots is w^* -null.

- (iii) There exists a bounded function $\pi : B_{X^*} \rightarrow \mathbb{R}$ such that

$$\pi(x^*) \geq \text{ca}(x_n^*) + \liminf_{n \rightarrow \infty} \pi(x^* + x_n^*)$$

whenever $x^* \in B_{X^*}$, $x^* + x_n^* \in B_{X^*}$ and x_1^*, x_2^*, \dots is w^* -null.

- (iv) We have $\pi_X^\alpha < \infty$ on B_{X^*} for every $\alpha < \omega_1$ where $\pi_X^0(x^*) = 0$,

$$\pi_X^{\alpha+1}(x^*) = \sup \left\{ \text{ca}(x_n^*) + \liminf_{n \rightarrow \infty} \pi_X^\alpha(x^* + x_n^*) : \right.$$

$$\left. x^* + x_n^* \in B_{X^*} \text{ and } x_n^* \xrightarrow{w^*} 0 \right\}$$

and

$$\pi_X^\beta(x^*) = \lim_{\alpha \nearrow \beta} \pi_X^\alpha(x^*) \quad \text{if } \beta \text{ is limit.}$$

Note that it is possible to replace “liminf” with “limsup” in the condition (ii). Of course, there is a version of (iv) based on μ instead of π , we prefer the current version nevertheless, as the sequence π_X^α is related to the Szlenk derivatives studied in [3]. It can be shown that a separable Banach space X has summable Szlenk index if and only if $\pi_X^\omega < \infty$ on B_{X^*} .

Proof (sketch). (i) \Rightarrow (ii): It is known that the function $\mu(x^*) = \|x^*\|$ has the desired property if X is isometric to a subspace of c_0 [17].

(ii) \Rightarrow (i): This is a consequence of [16, Theorem 3.3].

(ii) \Rightarrow (iii): The function $\pi = -2\mu$ works.

(iii) \Rightarrow (ii): The function $\mu = -\pi$ works.

(iii) \Rightarrow (iv): We can assume that $\pi \geq 0$, in which case $\pi_X^\alpha \leq \pi$.

(iv) \Rightarrow (iii): The function $\pi(x^*) = \lim_{\alpha \nearrow \omega_1} \pi_X^\alpha(x^*)$ works. \square

Now, we proceed to our problem.

Question 7.3. *Does there exist some $\alpha < \omega_1$ such that $\pi_X^\alpha = \pi_X^{\alpha+1}$ for every Banach space X isomorphic to c_0 ?*

By a classical result of W. B. Johnson and M. Zippin [13], a subspace of c_0 is isomorphic to c_0 if and only if it is an \mathcal{L}_∞ -space. Therefore, if such $\alpha < \omega_1$ as in the question exists, then a separable Banach space X is isomorphic to c_0 if and only if it is an \mathcal{L}_∞ -space, $\pi_X^\alpha < \infty$ and $\pi_X^\alpha = \pi_X^{\alpha+1}$.

Considering results from Section 3, we expect that the answer is negative for the class of all spaces which are isomorphic to the c_0 -sum of a sequence of finite-dimensional spaces and, in particular, for the class of all spaces which can be embedded isomorphically into c_0 .

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DEPARTMENT OF MATHEMATICAL ANALYSIS, CHARLES UNIVERSITY, SOKOLOVSKÁ 83, 186 75 PRAGUE 8, CZECH REPUBLIC

E-mail address: kurka.ondrej@seznam.cz